

From Dimensional to Cut-Off Regularization *

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We extent the standard approach of dimensional regularization of Feynman diagrams: we replace the transition to lower dimensions by a 'natural' cut-off regulator. Introducing an external regulator of mass $(\lambda)^{2\epsilon}$, we regain in the limit $\epsilon \rightarrow 0$ and $\epsilon \geq 0$ the results of dimensional and cut-off regularization, respectively. We demonstrate the versatility and adequacy of the different regularization schemes for practical examples (such as non covariant regularization, the axial anomaly or regularization in effective field theories).

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Renormalization is a genuine feature of relativistic field theories: in the presence of the nontrivial physical vacuum all observables result from the renormalization of unobservable 'bare' properties, even in finite theories [1], [2]. Among different schemes [3], dimensional regularization (DR) [4], [5], [6], [7] of Feynman diagrams is rather popular: both its technical simplicity and the preservation of fundamental symmetries, like gauge invariance and Ward identities, renders it the most widely used scheme for practical calculations (a collection of the basic formulae is found in refs. [4], [8], [9]). However, with recent developments, such as the chiral perturbation theory, the issue of the proper regularization of effective field theories has gained new interest [10], [11].

The basic idea of dimensional regularization is to regularize Feynman integrals of arbitrary dimension d by lowering the dimensionality to $D = d - 2\epsilon$ (with $\epsilon > 0$) and to isolate infinities as singularities in the analytic continuation $\epsilon \rightarrow 0$; a regulator mass $(\mu^2)^\epsilon$ enters from the appropriate dimensional mass dependence of the diagram. Just to remind of the standard relations: a simple example

$$I_{DR}(m^2, \mu^2) = (\mu^2)^\epsilon \int d^{d-2\epsilon} k \frac{(k^2)^l}{(k^2 + m^2)^n}, \quad (1)$$

yields explicitly

$$I_{DR}(m^2, \mu^2) = \pi^{d/2-\epsilon} \left(\frac{\Gamma(l + \frac{d}{2} - \epsilon)}{\Gamma(\frac{d}{2} - \epsilon)} \frac{\Gamma(n + \epsilon - l - \frac{d}{2})}{\Gamma(n)} \left(\frac{\mu^2}{m^2} \right)^\epsilon \right) (m^2)^{\frac{d}{2}+l-n}. \quad (2)$$

In this note we would like to extent and explore DR,

such as for treating the axial anomaly [12], [13], its applicability to non covariant Feynman diagrams [14], [15], or for a direct relation to cut-off regularization [16] and the underlying physical scales, a regularization scheme, as frequently favored in phenomenological effective field theories.

Our main goal in this note is to extent dimensional regularization, calling it natural regularization (NR), which both preserves the simplicity and the technical formalism of DR and establishes simultaneously a direct link to cut-off regularization (CR). To motivate our generalization, we formally transform a regulated $d-2\epsilon$ dimensional Feynman integral back to the d -dimensional space upon introducing the new (dimensionless) variable

$$\frac{k^2}{m^2} = x^{2+\epsilon}; \quad \int d^{d-2\epsilon} k \left(\frac{k}{m} \right) = \int d^d x, \quad (3)$$

which preserves, upon introducing the regulator mass $(\mu^2)^\epsilon$ the results obtained above in DR. The direct relation to cut-off regularization is now established in modifying eq. (3) by a cut-off mass M such, that in the limit of M being equal to the mass scale m of the integral itself, i. e. $M = m$ DR is recovered, and at the same time provides a direct route to CR. Explicitly, we introduce a rather general regulating factor as

$$\int d^d k f(k^2, m^2) \rightarrow \int d^d k f(k^2, m^2) \left(\frac{\lambda^2}{k^2 + M^2} \right)^\epsilon, \quad (4)$$

where the scales λ and M are identified with the external scale parameters in the problem. Evidently, the extension above provides flexibility enough to combine dimensional and cut-off regularization: choosing in the previous equation

$$\lambda^2 = \mu^2, M^2 = m^2, \quad \text{with } \epsilon \rightarrow 0:$$

$$I^\epsilon(k^2, m^2, \mu^2) = (\mu^2)^\epsilon \int d^d k \frac{(k^2)^l}{(k^2 + m^2)^{n+\epsilon}}, \quad (5)$$

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yields the correct expression for natural or, equivalently, dimensional regularization, whereas

$$\lambda^2 = M^2 = \Lambda^2, \text{ with } \epsilon > 0 : \\ I^\epsilon(k^2, m^2, \Lambda^2) = \int d^d k \frac{(k^2)^l}{(k^2 + m^2)^n} \left(\frac{\Lambda^2}{k^2 + \Lambda^2} \right)^\epsilon \quad (6)$$

yields the standard representation for cut-off regularization (the normalization point for the cut-off can be chosen at will by replacing $\Lambda^2 \rightarrow (\Lambda')^2$ in the numerator of the regulator). The equivalence of the generalized formulae above to DR and CR is easy to show using appropriate integral representations [17]: the final result agrees with both limits, except of trivial constants, which may be removed in the subtraction scheme.

Deferring further details of our formalism to a forthcoming note (a comprehensive list of references is given there), we just collect the most important consequences and characteristics of the extension above :

- **Regularization of one-loop integrals in NR:**

Coming back to eq. (1) we obtain in NR

$$I_{NR}(m^2, \mu^2) = \pi^{d/2} \frac{\Gamma(l + \frac{d}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(n + \epsilon - l - \frac{d}{2})}{\Gamma(n + \epsilon)}$$

The result agrees with DR: the double singularity is extracted from $\Gamma(\epsilon)$ and from the integration (by parts) of the parameter β ;

- **Non covariant (split) regularization** [14], [19]:

NR allows to introduce an arbitrary number of splittings (regulators and mass scales) as

$$\left(\frac{\mu^2}{k^2 + m^2} \right)^\epsilon \rightarrow \left(\frac{\mu_1^2}{k^2 + m_1^2} \right)^{\epsilon - \epsilon_1} \left(\frac{\mu_2^2}{k^2 + m_2^2} \right)^{\epsilon_1 - \epsilon_2} \dots; \quad (11)$$

and thus a natural application to non covariant formulations, such as gauge theories or to regularization on the light cone, by either introducing various infinitesimal external parameters or by regularizing arbitrarily one integration variable with just one single infinitesimal parameter either before or (for factorized integrals) after introducing Feynman parametrization;

$$\left(\frac{\mu^2}{m^2} \right)^\epsilon (m^2)^{\frac{d}{2} + l - n}, \quad (7)$$

which differs from DR in eq. (2) only in irrelevant constants;

- **Consistency of $\int d^d k (k^2)^n = 0$:**

Regularizing naturally

$$\int d^d k (k^2)^n \left(\frac{\mu^2}{k^2 + m^2} \right)^\epsilon, \quad (8)$$

yields, upon identifying the arbitrary mass scale m with the only scale in the integral, the regulator mass, i. e. $m^2 = \mu^2$, a constant piece, which is removed in the subtraction scheme;

- **Multi-loop integrals** [18]:

Again we provide only one classical example, i. e. the ‘setting sun’ diagram (in 4 dimensions)

$$I(m^2, p^2) = \int \int d^d k d^d q \frac{1}{(k^2 + m^2)(q^2 + m^2)((q - k + p)^2 + m^2)}, \quad (9)$$

which immediately yields after Feynman parametrization and upon performing the regularized integrals in momentum space

$$I(m^2, p^2) = (\pi^2)^2 \mu^2 \frac{\Gamma(\epsilon)}{\Gamma(2 + \epsilon)} \int_0^1 d\alpha \int_0^1 d\beta \frac{\beta^{\epsilon-1}(1-\beta)}{\{\alpha(1-\alpha)\beta(1-\beta)p^2 + m^2 [\alpha(1-\alpha)\beta + (1-\beta)]\}^\epsilon}. \quad (10)$$

- **Gauge invariance** [4]:

to recover gauge invariance (which is lost in going over from DR to NR by the replacement $d - 2\epsilon \rightarrow d$), we remind that we have two scales in any integral

$$\begin{aligned} \int d^d k \frac{(k^2)^l}{(k^2 + m^2)^n} &\rightarrow \int d^d k \frac{(k^2)^l}{(k^2 + m^2)^n} \left(\frac{\mu^2}{k^2 + m^2} \right)^\epsilon \\ &\leftrightarrow \int d^d k \left(\frac{\mu^2}{k^2} \right)^\epsilon \frac{(k^2)^l}{(k^2 + m^2)^n}, \end{aligned} \quad (12)$$

which recovers gauge invariance upon an appropriate combination of the two regularization modes;

- **axial anomaly** [2], [3], [20], [21] :

As well known, ‘standard’ dimensional regularization faces problems for the axial anomaly, as the definition of the pseudo-scalar matrix $\gamma_5 =$

$i\epsilon_{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta$ via the antisymmetric ϵ -tensor is genuinely related to 4 dimensions; technically, the problem is overcome by formal extensions of γ_5 to $d > 4$, defining appropriate commutation and anti-commutation relations with the matrices γ_μ .

NR provides a consistent regularization: upon Feynman parametrizing and regularizing the axial triangle yields

$$T_{\mu\nu\lambda} = -i \int d\alpha \int d\beta \int d^4k \frac{(\mu^2)^\epsilon \text{Tr}[(\not{k} + \not{k}_1 + m)\gamma_\mu(\not{k} + m)\gamma_\nu(\not{k} - \not{k}_2 + m)\gamma_\lambda\gamma_5]}{\{\alpha[(k+k_1)^2 - m^2] + \beta(k^2 - m^2) + (1-\alpha-\beta)[(k-k_2)^2 - m^2]\}^{3+\epsilon}}. \quad (13)$$

The convergent loop integral is now readily performed after an appropriate shift in the loop momentum;

• **Optimal subtraction scheme:**

From the results above, we realize that NR provides much flexibility for different regularization procedures by an appropriate superposition of the different prescriptions, such as

$$I(m^2) \rightarrow \lambda \int d^4k \frac{k^2}{(k^2 + m^2)^{1+\epsilon}} + (1-\lambda) \int d^4k \frac{(k^2)^{1+\epsilon}}{k^2 + m^2}, \quad (14)$$

with an arbitrary constant λ . This flexibility in the regularization might be either used to improve the convergence of a perturbative expansion or, alternatively, these additional constants have no physical content and thus can be completely removed (along the \overline{MS} subtraction scheme [2]);

• **Integrating out effective field theories:**

Already a simple example exhibits some shortcomings of DR in context with effective field theories: the well defined integral in dimensional regularization

$$I_{DR}(m, \epsilon) = (\mu^2)^\epsilon \int d^{4-2\epsilon}k \frac{1}{(k^2 + m^2)} \sim m^2 \ln\left(\frac{\mu^2}{m^2}\right), \quad (15)$$

vanishes identically according to eq. (8), if expanded around the mass parameter m

$$I_{DR}(m, \epsilon)^{EFT} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(m^2)^{n+1}} \int d^{4-2\epsilon}k (k^2)^n = 0. \quad (16)$$

Natural regularization yields a finite result, upon expanding the denominator and regularizing it with the regulator mass μ .

As an extension of this example we integrate out masses in effective field theories. Explicitly we find for

$$I(m^2, M^2) = \int d^4k \frac{1}{(k^2 + m^2)} \frac{1}{(k^2 + M^2)}$$

$$= \pi^2 \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \frac{1}{(M^2 - m^2)} \left(m^2 \ln \frac{m^2}{\mu^2} - M^2 \ln \frac{M^2}{\mu^2} \right). \quad (17)$$

This standard result is recovered in NR, upon expanding the denominators symmetrically around M and m and after regularizing via an effective regulator scale,

$$\begin{aligned} I(m^2, M^2) &= \int d^4k \frac{1}{(k^2 + m^2)} \frac{1}{k^2 + M^2} \left(\frac{\mu^2}{k^2 + M^2} \right)^\epsilon \\ &= -\sum \left(-\frac{1}{M^2} \right)^{n+1} \int d^4k \frac{(k^2)^n}{(k^2 + m^2)} \left(\frac{\mu^2}{k^2 + m^2} \right)^\epsilon \\ &= \pi^2 \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \frac{m^2}{(M^2 - m^2)} \ln \frac{m^2}{\mu^2} + (M \leftrightarrow m); \end{aligned} \quad (18)$$

• **Unifying DR and CR:**

NR provides a transparent unification of both schemes: upon introducing 2 independent external parameters λ, M for the regularization, we obtain for a characteristic example (compare eq. (4))

$$\begin{aligned} J(\lambda, M, \epsilon) &= \int d^3k \frac{k^2}{k^2 + m^2} \left(\frac{\lambda^2}{k^2 + M^2} \right)^\epsilon \\ &= \pi^{3/2} \frac{3}{2} \frac{\Gamma(-\frac{3}{2} + \epsilon)}{\epsilon \Gamma(\epsilon)} \left(\frac{\lambda^2}{m^2} \right)^\epsilon m^3 \\ &\quad F\left(\epsilon - \frac{3}{2}, \epsilon, \epsilon + 1, -\frac{M^2 - m^2}{m^2}\right), \end{aligned} \quad (19)$$

(with the hypergeometric function $F(a, b, c, z)$), which holds for all values of ϵ . Evaluating F , we immediately obtain, for example, in

$$\begin{aligned} DR : \lambda = \mu; M = m : \epsilon \rightarrow 0; \\ CR(Dipole) : \lambda = M = \Lambda : \epsilon = 2; \end{aligned} \quad (20)$$

furthermore, similarly as in split regularization, we may introduce in CR different discrete or continuous external cut-off scales both in Λ and ϵ , such as

$$\sim \int J(\Lambda, \epsilon) \rho(\Lambda, \epsilon) d\Lambda d\epsilon \quad (21)$$

(with the density distribution $\rho(\Lambda, \epsilon)$ appropriately normalized) ;

• **Functional relation of a cut-off and regulator mass in CR, DR:**

Though a rigorous one-to-one comparison explicitly depends on the diagrams under consideration, the global relation is readily extracted for the two most frequent examples, isolating a quadratic or logarithmic divergence in NR in 4 (in general: d) dimensional Euclidean space, upon evaluating the combination of integrals

$$I_1^\epsilon(m^2, \mu^2) - m^2 I_2^\epsilon(m^2, \mu^2) \quad (22)$$

for

$$I_n^\epsilon(m^2, \mu^2) = \int d^4k \frac{\mu^2}{(k^2 + m^2)^{n+\epsilon}} \quad (23)$$

both in dimensional and cut-off regularization. Comparing the limit $\epsilon \rightarrow 0$, with cut-off regularization for a scale mass Λ yields

$$\frac{\Lambda^4}{\Lambda^2 + m^2} \sim \Lambda^2 = -\frac{2}{\epsilon} \left(\frac{\mu^2}{m^2} \right)^\epsilon m^2. \quad (24)$$

for $\Lambda \rightarrow \infty$ (and $\epsilon \rightarrow 0$);

• **Sharp momentum cut-off versus soft form factor:**

For integrals with a vanishing mass scale, DR according to eq. (8) is in apparent contrast to CR, yielding for a sharp cut-off Λ

$$I_C^{(n)}(\Lambda) = \int_0^\Lambda d^4k (k^2)^n = \pi^2 \frac{(\Lambda^2)^{n+2}}{n+2}, \quad (25)$$

In NR we obtain the same result, upon introducing a regulator mass μ for a finite ϵ , i. e.

$$\begin{aligned} I_{NR}^{(n)} &= \int d^4k (k^2)^n \frac{(\mu^2)^\epsilon}{(k^2 + \mu^2)^\epsilon} \\ &= \pi^2 \frac{\Gamma(n+2)}{\Gamma(2)} \frac{\Gamma(\epsilon - (n+2))}{\Gamma(\epsilon)} (\mu^2)^{n+2}, \end{aligned} \quad (26)$$

which evidently agrees with cut-off regularization upon identifying $\Lambda = \mu$ for $\epsilon = n+3$ (reflecting the asymptotic behavior of the integral like a monopole $\sim 1/k^2$). Clearly, for effective field theories this form of regularization is appealing, as it explicitly reflects the typical scale of the underlying physics, i. e. inverse size of the effective degrees of freedom.

The example above holds also for cut-offs via soft form factors: a quantitative relation between a sharp and a soft cut-off is readily established such as for

$$I_{SC}(m^2, \Lambda^2) = \int_0^\Lambda d^4k \frac{(k^2)^\ell}{(k^2 + m^2)^n}, \quad (27)$$

upon introducing the transformations

$$q^2 = \frac{\Lambda^2}{\Lambda^2 - k^2} k^2 \longleftrightarrow k^2 = \frac{\Lambda^2}{\Lambda^2 + q^2} \cdot q^2, \quad (28)$$

ending up with

$$I_{SC}(m^2, \Lambda^2) = \frac{(\Lambda^2)^\ell}{(\Lambda^2 + m^2)^n} \int d^4q \left(\frac{\Lambda^2}{\Lambda^2 + q^2} \right)^3 \left(\frac{q^2}{q^2 + \Lambda^2} \right)^\ell \left(\frac{q^2 + \Lambda^2}{q^2 + \frac{\Lambda^2}{\Lambda^2 + m^2} m^2} \right)^n. \quad (29)$$

The result clearly demonstrates the equivalence of sharp and soft cut-offs: independent of the explicit momentum dependence of the integral under investigation, sharp cut-off regularization in 4 (d in general) dimensions is strictly equivalent to a soft cubic form-factor, guaranteeing a strict convergence of the integral considered;

• **Incorporating external scales in effective**

field theories [10], [11], [22], [23], [24]:

Effective field theories, such as Chiral Perturbation (χ PT) or Heavy Baryon Effective Theory (HBETH) are, as expansions in powers of the internal momentum scale, in the strict sense non-renormalizable field theories; as a consequence the transition to effective objects introduces typical scales of their spatial extension. The origin of the problem is the very

nature of diverging diagrams in NR (DR), yielding just from power counting (compare eqs. (1,2))

$$I_{NR}^n = \int d^d k \frac{(k^2)^l}{(k^2 + m^2)^n} \sim (m^2)^{\frac{d}{2}+l-n} \ln\left(\frac{m^2}{\mu^2}\right). \quad (30)$$

Evidently, the only physical scale in NR (DR) is the mass m of the intermediate particle, which sets the scale just for any momentum dependence of the integrals considered.

Already a very simple example strikingly exhibits the consequences in practical calculations: comparing the self energy correction of, say, a baryon, induced on the one-loop level by the exchange of mesons with masses m , m' , respectively. In DR their contribution scales qualitatively with the ratio

$$\frac{\Sigma_{m'}}{\Sigma_m} \sim \left(\frac{m'}{m}\right)^{\frac{d}{2}+l-n}, \quad (31)$$

Thus for $m' \gg m$ and for the momentum expansion in effective field theories with typically $\frac{d}{2}+l \gg n$, in a perturbative loop expansion heavy meson exchange (with mass m') dominates. This contradicts an intuitive physical understanding, for example from the uncertainty principle, from which $\Sigma' \ll \Sigma$ is expected. Only introducing for example external form factors guarantees the right scales of the corresponding self energies.

We summarize briefly the main findings in our short note. In NR, our starting point is — compared to dimensional regularization — slightly different: while DR enforces convergence of diverging integrals by dimensionally reducing the corresponding phase space, NR cuts down the high momentum components by a dimensionless regulator strictly in 4 (in general: in d) dimensions. Without losing the technical simplicity of DR, this simple modification introduces the regulator scale in a very natural way: the regularization itself allows a systematic and well-defined finite extension to singular diagrams of arbitrary order, by providing a direct route from DR to cut-off regularization. While DR leaves the internal mass scale m of integration in momentum space unchanged, NR allows for the introduction of external mass scales, for example via conventional monopole or dipole form factors, as characteristic for phenomenological low energy nuclear physics (for further details and quantitative

insight into these and related questions we refer to a comprehensive forthcoming article).

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